Cohomology of the Adjoint of Hopf Algebras

J. Scott Carter*
University of South Alabama

Alissa S. Crans
Loyola Marymount University

Mohamed Elhamdadi
University of South Florida

Masahico Saito†
University of South Florida

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Abstract

A cohomology theory of the adjoint of Hopf algebras, via deformations, is presented by means of diagrammatic techniques. Explicit calculations are provided in the cases of group algebras, function algebras on groups, and the bosonization of the super line. As applications, solutions to the YBE are given and quandle cocycles are constructed from groupoid cocycles.

1 Introduction

Algebraic deformation theory [10] can be used to define 2-dimensional cohomology in a wide variety of contexts. This theory has also been understood diagrammatically [7, 16, 17] via PROPs, for example. In this paper, we use diagrammatic techniques to define a cohomological deformation of the adjoint map $\mathrm{ad}(x \otimes y) = \sum S(y_{(1)}) x y_{(2)}$ in an arbitrary Hopf algebra. We have concentrated on the diagrammatic versions here because diagrammatics have led to topological invariants [6, 13, 19], diagrammatic methodology is prevalent in understanding particle interactions and scattering in the physics literature, and most importantly kinesthetic intuition can be used to prove algebraic identities.

The starting point for this calculation is a pair of identities that the adjoint map satisfies and that are sufficient to construct Woronowicz's solution [22] $R = (1 \otimes \mathrm{ad})(\tau \otimes 1)(1 \otimes \Delta)$ to the Yang-Baxter equation (YBE): $(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R)$. We use deformation theory to define an extension 2-cocycle. Then we show that the resulting 2-coboundary map, when composed with the Hochschild 1-coboundary map is trivial. A 3-coboundary is defined via the "movie move" technology. Applications of this cohomology theory include constructing new solutions to the YBE by deformations and constructing quandle cocycles from groupoid cocycles that arise from this theory.

The paper is organized as follows. Section 2 reviews the definition of Hopf algebras, defines the adjoint map, and illustrates Woronowicz's solution to the YBE. Section 3 contains the deformation theory. Section 4 defines the chain groups and differentials in general. Example calculations

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in the case of a group algebra, the function algebra on a group, and a calculation of the 1- and 2-dimensional cohomology of the bosonization of the superline are presented in Section 5. Interestingly, the group algebra and the function algebra on a group are cohomologically different. Moreover, the conditions that result when a function on the group algebra satisfies the cocycle condition coincide with the definition of groupoid cohomology. This relationship is given in Section 6, along with a construction of quandle 3-cocycles from groupoid 3-cocycles. In Section 7, we use the deformation cocycles to construct solutions to the Yang-Baxter equation.

1.1 Acknowledgements

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2 Preliminaries

We begin by recalling the operations and axioms in Hopf algebras, and their diagrammatic conventions depicted in Figures 1 and 2.

A *coalgebra* is a vector space C over a field k together with a *comultiplication* $\Delta : C \to C \otimes C$ that is bilinear and *coassociative*: $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$. A coalgebra is *cocommutative* if the comultiplication satisfies $\tau\Delta = \Delta$, where $\tau : C \otimes C \to C \otimes C$ is the transposition $\tau(x \otimes y) = y \otimes x$. A *coalgebra with counit* is a coalgebra with a linear map called the *counit* $\epsilon : C \to k$ such that $(\epsilon \otimes 1)\Delta = 1 = (1 \otimes \epsilon)\Delta$ via $k \otimes C \cong C$. A *bialgebra* is an algebra A over a field k together with a linear map called the *unit* $\eta : k \to A$, satisfying $\eta(a) = a\mathbf{1}$ where $\mathbf{1} \in A$ is the multiplicative identity and with an associative multiplication $\mu : A \otimes A \to A$ that is also a coalgebra such that the comultiplication Δ is an algebra homomorphism. A *Hopf algebra* is a bialgebra C together with a map called the *antipode* $S : C \to C$ such that $\mu(S \otimes 1)\Delta = \eta\epsilon = \mu(1 \otimes S)\Delta$, where ϵ is the counit.

In diagrams, the compositions of maps are depicted from bottom to top. Thus a multiplication μ is represented by a trivalent vertex with two bottom edges representing $A \otimes A$ and one top edge representing A. Other maps in the definition are depicted in Fig. 1 and axioms are depicted in Fig. 2.

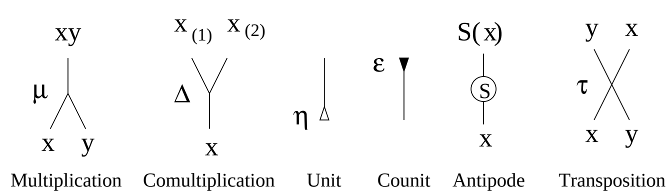

Figure 1: Operations in Hopf algebras

Let H be a Hopf algebra. The adjoint map $\mathrm{Ad}_y : H \to H$ for any $y \in H$ is defined by $\mathrm{Ad}_y(x) = S(y_{(1)})xy_{(2)}$, where we use the common notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$ and $\mu(x \otimes y) = xy$.

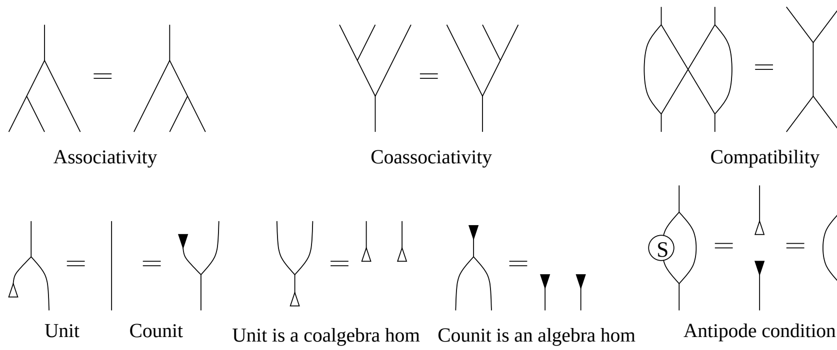

Figure 2: Axioms of Hopf algebras

Its diagram is depicted in Fig. 3. Notice the analogy with group conjugation: in a group ring $H = kG$ over a field k, where $\Delta(y) = y \otimes y$ and $S(y) = y^{-1}$, we have $\mathrm{Ad}_y(x) = y^{-1}xy$.

When we view the adjoint map as a map from $H \otimes H$ to H, we use the notation

$$\mathrm{ad} : H \otimes H \to H, \quad \mathrm{ad}(x \otimes y) = \mathrm{Ad}_y(x).$$

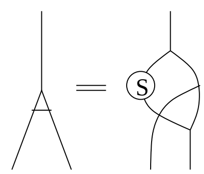

Figure 3: Adjoint map in a Hopf algebra

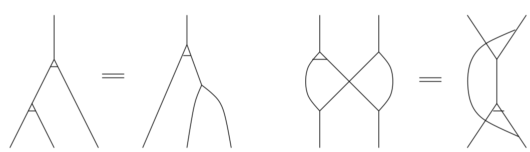

Figure 4: Conditions for the YBE for Hopf algebras

Definition 2.1 Let H be a Hopf algebra and ad be the adjoint map. Then the linear map $R_{\mathrm{ad}} : H \otimes H \to H \otimes H$ defined by

$$R_{\mathrm{ad}} = (1 \otimes \mathrm{ad})(\tau \otimes 1)(1 \otimes \Delta)$$

is said to be the R-matrix *induced from* ad.

Lemma 2.2 *The R-matrix induced from* ad *satisfies the YBE.*

Proof. In Fig. 5, it is indicated that the YBE follows from two properties of the adjoint map:

$$\mathrm{ad}(\mathrm{ad} \otimes 1) = \mathrm{ad}(1 \otimes \mu) \quad \text{and} \tag{1}$$

$$(\mathrm{ad} \otimes \mu)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta) = (1 \otimes \mu)(\tau \otimes 1)(1 \otimes \Delta)(1 \otimes \mathrm{ad})(\tau \otimes 1)(1 \otimes \Delta). \tag{2}$$

It is known that these properties are satisfied, and proofs are found in [11, 22]. Here we include diagrammatic proofs for reader's convenience in Fig. 6 and Fig. 7, respectively. □

Definition 2.3 We call the above equalities (1) and (2) the *adjoint conditions*.

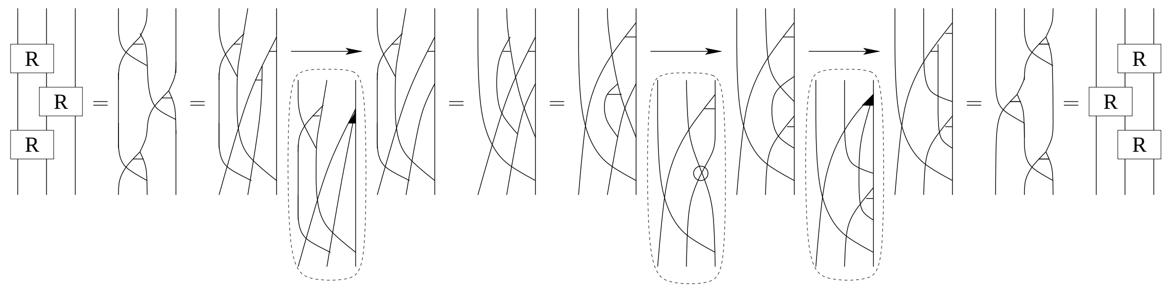

Figure 5: YBE by the adjoint map

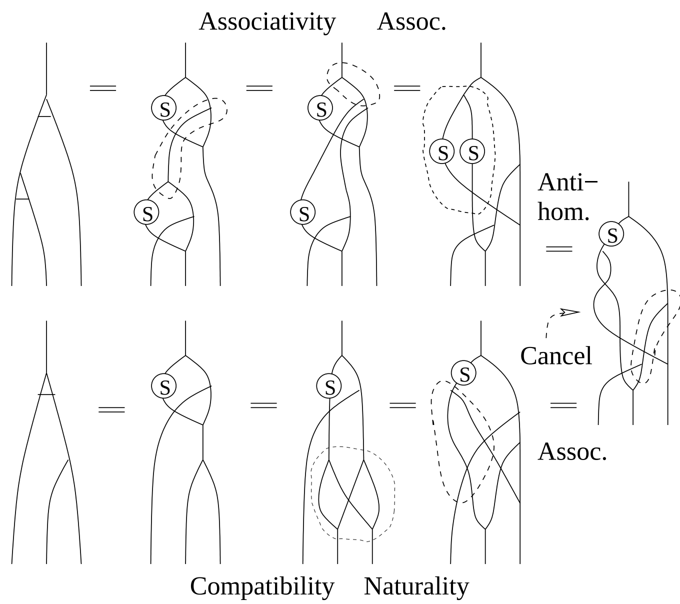

Figure 6: Adjoint condition (1)

Remark 2.4 The equality (1) is equivalent to the fact that the adjoint map defines an algebra action of H on itself (see [14]). Specifically, $(a \triangleleft b) \triangleleft c = a \triangleleft (bc)$ for any $a, b, c \in H$, where $\triangleleft$ denotes the right action defined by the adjoint: $a \triangleleft b = \mathrm{ad}(a \otimes b)$. The equality (2) can be similarly rewritten as:

$$a_{(1)} \triangleleft b_{(1)} \otimes a_{(2)} b_{(2)} = (a \triangleleft b_{(2)})_{(1)} \otimes b_{(1)} (a \triangleleft b_{(2)})_{(2)}.$$

Remark 2.5 It was pointed out to us by Sommerhaeuser that the induced R-matrix R_{ad} is invertible with inverse

$$R_{\mathrm{ad}}^{-1}(b \otimes a) = b_{(3)} a S^{-1}(b_{(2)}) \otimes b_{(1)}.$$

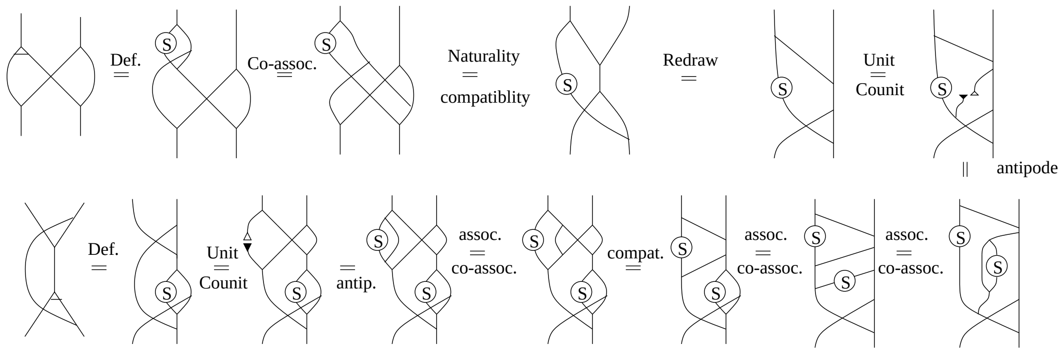

Figure 7: Adjoint condition (2)

3 Deformations of the Adjoint Map

We follow the exposition in [16] for deformation of bialgebras to propose a similar deformation theory for the adjoint map. In light of Lemma 2.2, we deform the two equalities (1) and (2). Let H be a Hopf algebra and ad its adjoint map.

Definition 3.1 A deformation of (H, ad) is a pair (H_t, ad_t) where H_t is a $k[[t]]$-Hopf algebra given by $H_t = H \otimes k[[t]]$ with all Hopf algebra structures inherited by extending those on H_t with the identity on the $k[[t]]$ factor (the trivial deformation as a Hopf algebra), with a deformations of ad given by $\mathrm{ad}_t = \mathrm{ad} + t\mathrm{ad}_1 + \cdots + t^n \mathrm{ad}_n + \cdots : H_t \otimes H_t \to H_t$ where $\mathrm{ad}_i : H \otimes H \to H$, $i = 1, 2, \cdots$, are maps.

Suppose $\bar{\mathrm{ad}} = \mathrm{ad} + \cdots + t^n \mathrm{ad}_n$ satisfies the adjoint conditions (equalities (1) and (2)) mod t^{n+1}, and suppose that there exist $\mathrm{ad}_{n+1} : H \otimes H \to H$ such that $\bar{\mathrm{ad}} + t^{n+1}\mathrm{ad}_{n+1}$ satisfies the adjoint conditions mod t^{n+2}. Define $\xi_1 \in \mathrm{Hom}(H^{\otimes 3}, H)$ and $\xi_2 \in \mathrm{Hom}(H^{\otimes 2}, H^{\otimes 2})$ by

$$\begin{aligned}
\bar{\mathrm{ad}}(\bar{\mathrm{ad}} \otimes 1) - \bar{\mathrm{ad}}(1 \otimes \mu) &= t^{n+1}\xi_1 \mod t^{n+2}, \\
(\bar{\mathrm{ad}} \otimes \mu)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)& \\
-(1 \otimes \mu)(\tau \otimes 1)(1 \otimes \Delta)(1 \otimes \bar{\mathrm{ad}})(\tau \otimes 1)(1 \otimes \Delta) &= t^{n+1}\xi_2 \mod t^{n+2}.
\end{aligned}$$

For the first adjoint condition (1) of $\bar{\mathrm{ad}} + t^{n+1}\mathrm{ad}_{n+1}$ mod t^{n+2} we obtain:

$$(\bar{\mathrm{ad}} + t^{n+1}\mathrm{ad}_{n+1})((\bar{\mathrm{ad}} + t^{n+1}\mathrm{ad}_{n+1}) \otimes 1) - (\bar{\mathrm{ad}} + t^{n+1}\mathrm{ad}_{n+1})(1 \otimes \mu) = 0 \mod t^{n+2}$$

which is equivalent by degree calculations to:

$$\mathrm{ad}(\mathrm{ad}_{n+1} \otimes 1) + \mathrm{ad}_{n+1}(\mathrm{ad} \otimes 1) - \mathrm{ad}_{n+1}(1 \otimes \mu) = \xi_1.$$

For the second adjoint condition (2) of $\bar{\mathrm{ad}} + t^{n+1}\mathrm{ad}_{n+1}$ mod t^{n+2} we obtain:

$$\begin{aligned}
&((\bar{\mathrm{ad}} + t^{n+1}\mathrm{ad}_{n+1}) \otimes \mu)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta) \\
-\ &(1 \otimes \mu)(\tau \otimes 1)(1 \otimes \Delta)(1 \otimes (\bar{\mathrm{ad}} + t^{n+1}\mathrm{ad}_{n+1}))(\tau \otimes 1)(1 \otimes \Delta) \\
=\ &0 \mod t^{n+2}
\end{aligned}$$

which is equivalent by degree calculations to:

$$\begin{aligned} & (\mathrm{ad}_{n+1} \otimes \mu)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta) \\ - & \quad (1 \otimes \mu)(\tau \otimes 1)(1 \otimes \Delta)(1 \otimes \mathrm{ad}_{n+1})(\tau \otimes 1)(1 \otimes \Delta) \quad = \quad \xi_2. \end{aligned}$$

In summary we proved the following:

Lemma 3.2 *The map* $\bar{\mathrm{ad}} + t^{n+1}\mathrm{ad}_{n+1}$ *satisfies the adjoint conditions mod* t^{n+2} *if and only if*

$$\begin{aligned} \mathrm{ad}(\mathrm{ad}_{n+1} \otimes 1) + \mathrm{ad}_{n+1}(\mathrm{ad} \otimes 1) - \mathrm{ad}_{n+1}(1 \otimes \mu) &= \xi_1, \\ \text{and} \quad (\mathrm{ad}_{n+1} \otimes \mu)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta) & \\ -(1 \otimes \mu)(\tau \otimes 1)(1 \otimes \Delta)(1 \otimes \mathrm{ad}_{n+1})(\tau \otimes 1)(1 \otimes \Delta) &= \xi_2. \end{aligned}$$

4 Differentials and Cohomology

4.1 Chain Groups

We define chain groups, for positive integers n, $n > 1$, and $i = 1, \ldots, n$ by:

$$\begin{aligned} C^{n,i}_{\mathrm{ad}}(H;H) &= \mathrm{Hom}(H^{\otimes(n+1-i)}, H^{\otimes i}), \\ C^{n}_{\mathrm{ad}}(H;H) &= \oplus_{i>0,\ i \leq n+1-i}\, C^{n,i}_{ad}(H;H). \end{aligned}$$

Specifically, chain groups in low dimensions of our concern are:

$$\begin{aligned} C^{2}_{\mathrm{ad}}(H;H) &= \mathrm{Hom}(H^{\otimes 2}, H), \\ C^{3}_{\mathrm{ad}}(H;H) &= \mathrm{Hom}(H^{\otimes 3}, H) \oplus \mathrm{Hom}(H^{\otimes 2}, H^{\otimes 2}). \end{aligned}$$

For $n = 1$, define

$$C^{1}_{\mathrm{ad}}(H;H) = \{ f \in \mathrm{Hom}_k(H,H) \mid f\mu = \mu(f \otimes 1) + \mu(1 \otimes f),\ \Delta f = (f \otimes 1)\Delta + (1 \otimes f)\Delta \ \}.$$

In the remaining sections we will define differentials that are homomorphisms between the chain groups:

$$d^{n,i} : C^{n}_{\mathrm{ad}}(H;H) \to C^{n+1,i}_{\mathrm{ad}}(H;H) (= \mathrm{Hom}(H^{\otimes(n+2-i)}, H^{\otimes i}))$$

that will be defined individually for $n = 1, 2, 3$ and for i with $2i \leq n+1$, and

$$\begin{aligned} D_1 &= d^{1,1} : C^{1}_{\mathrm{ad}}(H;H) \to C^{2}_{\mathrm{ad}}(H;H), \\ D_2 &= d^{2,1} + d^{2,2} : C^{2}_{\mathrm{ad}}(H;H) \to C^{3}_{\mathrm{ad}}(H;H), \\ D_3 &= d^{3,1} + d^{3,2} + d^{3,3} : C^{3}_{\mathrm{ad}}(H;H) \to C^{3}_{\mathrm{ad}}(H;H). \end{aligned}$$

4.2 First Differentials

By analogy with the differential for multiplication, we make the following definition:

Definition 4.1 The first differential

$$d^{1,1} : C^{1}_{\mathrm{ad}}(H;H) \to C^{2,1}_{\mathrm{ad}}(H;H)$$

is defined by

$$d^{1,1}(f) = \mathrm{ad}(1 \otimes f) - f\mathrm{ad} + \mathrm{ad}(f \otimes 1).$$

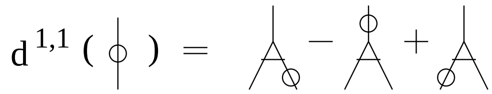

Figure 8: The 1-differential

Diagrammatically, we represent $d^{1,1}$ as depicted in Fig. 8, where a 1-cochain is represented by a circle on a string.

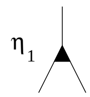

Figure 9: A diagram for a 2-cochain

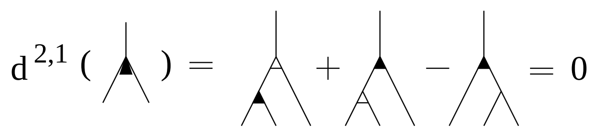

Figure 10: The 2-cocycle condition, Part I

4.3 Second Differentials

Definition 4.2 Define the second differentials by:

$$
\begin{aligned}
d^{2,1}_{\mathrm{ad}}(\phi) &= \mathrm{ad}(\phi \otimes 1) + \phi(\mathrm{ad} \otimes 1) - \phi(1 \otimes \mu), \\
d^{2,2}_{\mathrm{ad}}(\phi) &= (\phi \otimes \mu)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta) - (1 \otimes \mu)(\tau \otimes 1)(1 \otimes \Delta)(1 \otimes \phi)(\tau \otimes 1)(1 \otimes \Delta).
\end{aligned}
$$

Diagrams for 2-cochain and 2-differentials are depicted in Fig. 9, 10, and 11, respectively.

Theorem 4.3 $D_2 D_1 = 0$.

Proof. This follows from direct calculations, and can be seen from diagrams in Figs. 12 and 13. ☐

4.4 Third Differentials

Definition 4.4 We define 3-differentials as follows. Let $\xi_i \in C^{3,i}(H;H)$ for $i = 1, 2$. Then

$$
\begin{aligned}
d^{3,1}_{\mathrm{ad}}(\xi_1, \xi_2) &= \mathrm{ad}(\xi_1 \otimes 1) + \xi_1(1 \otimes \mu \otimes 1) - \xi_1(\mathrm{ad} \otimes 1^2 + 1^2 \otimes \mu), \\
d^{3,2}_{\mathrm{ad}}(\xi_1, \xi_2) &= (\mathrm{ad} \otimes \mu)(1 \otimes \tau \otimes 1)(1^2 \otimes \Delta)(\xi_2 \otimes 1) + (1 \otimes \mu)(\tau \otimes 1)(1 \otimes \xi_2)(R_{\mathrm{ad}} \otimes 1) \\
&\quad + (1 \otimes \mu)(1^2 \otimes \mu)(\tau \otimes 1^2)(1 \otimes \tau \otimes 1)(1^2 \otimes \Delta)((1^2 \otimes \xi_1) \\
&\qquad \cdot (1 \otimes \tau \otimes 1^2)(\tau \otimes 1^3)(1^2 \otimes \tau \otimes 1)(1 \otimes \Delta \otimes \Delta)) \\
&\quad - (\xi_1 \otimes \mu)(1^2 \otimes \tau \otimes 1)(1^2 \otimes \mu \otimes 1^2)(1 \otimes \tau \otimes 1^3)(\Delta \otimes \Delta \otimes \Delta) - \xi_2(1 \otimes \mu),
\end{aligned}
$$

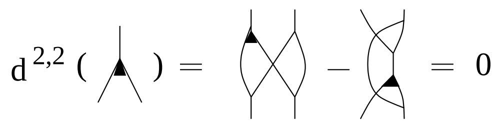

Figure 11: The 2-cocycle condition, Part II

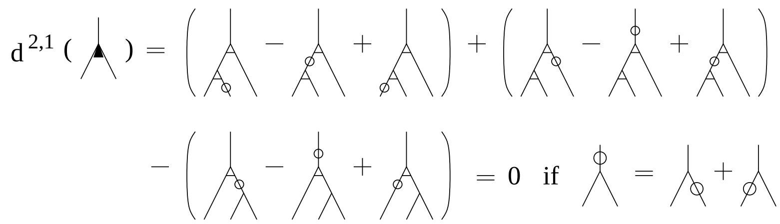

Figure 12: The 2-cocycle condition for a 2-coboundary, Part I

$$\begin{aligned} d^{3,3}_{\rm ad}(\xi_1,\xi_2) &= (1\otimes\mu\otimes 1)(\tau\otimes 1^2)(1\otimes\Delta\otimes 1)(1\otimes\xi_2)(\tau\otimes 1)(1\otimes\Delta) \\ &\quad +(\xi_2\otimes\mu)(1\otimes\tau\otimes 1)(\Delta\otimes\Delta)-(1\otimes\Delta)\xi_2. \end{aligned}$$

Diagrams for 3-cochains are depicted in Fig. 14. See Figs. 15, 16, and 17 for the diagrammatics for $d^{3,1}$, $d^{3,2}$ and $d^{3,3}$, respectively.

Theorem 4.5 *$D_3 D_2 = 0$.*

Proof. The proof follows from direct calculations that are indicated in Figs. 18, 19 and 20. We demonstrate how to recover algebraic calculations from these diagrams for the part $(d^{3,3}d^{2,2})(\eta_1)=0$ for any $\eta_1\in C^2(H;H)$. This is indicated in Fig. 20, where subscripts ad are suppressed for simplicity. Let $\xi_2=d^{2,2}(\eta_1)\in C^{3,2}(H;H)$ (note that $\xi_1=d^{2,1}(\eta_1)\in C^{3,1}(H;H)$ does not land in the domain of $d^{3,3}$). The first line of Fig. 20 represents the definition of the differential

$$\begin{aligned} d^{3,3}_{\rm ad}(\xi_1,\xi_2) &= (1\otimes\mu\otimes 1)(\tau\otimes 1^2)(1\otimes\Delta\otimes 1)(1\otimes\xi_2)(\tau\otimes 1)(1\otimes\Delta) \\ &\quad +(\xi_2\otimes\mu)(1\otimes\tau\otimes 1)(\Delta\otimes\Delta)-(1\otimes\Delta)\xi_2 \end{aligned}$$

where each term represents each connected diagram. The first parenthesis of the second line represents that

$$\xi_2=d^{2,2}(\eta_1) = (\eta_1\otimes\mu)(1\otimes\tau\otimes 1)(\Delta\otimes\Delta)-(1\otimes\mu)(\tau\otimes 1)(1\otimes\Delta)(1\otimes\eta_1)(\tau\otimes 1)(1\otimes\Delta)$$

is substituted in the first term

$$(1\otimes\mu\otimes 1)(\tau\otimes 1^2)(1\otimes\Delta\otimes 1)(1\otimes\xi_2)(\tau\otimes 1)(1\otimes\Delta).$$

When these two maps are applied to a general element $x\otimes y\in H\otimes H$, the results are computed as

$$\begin{aligned} &\eta_1(x_{(1)}\otimes y_{(2)(1)})_{(1)}\otimes y_{(1)}\eta_1(x_{(1)}\otimes y_{(2)(1)})_{(2)}\otimes x_{(2)}y_{(2)(2)}, \\ &-\eta_1(x\otimes y_{(2)(2)})_{(1)(1)}\otimes y_{(1)}\eta_1(x\otimes y_{(2)(2)})_{(1)(2)}\otimes y_{(2)(1)}\eta_1(x\otimes y_{(2)(2)})_{(2)}. \end{aligned}$$

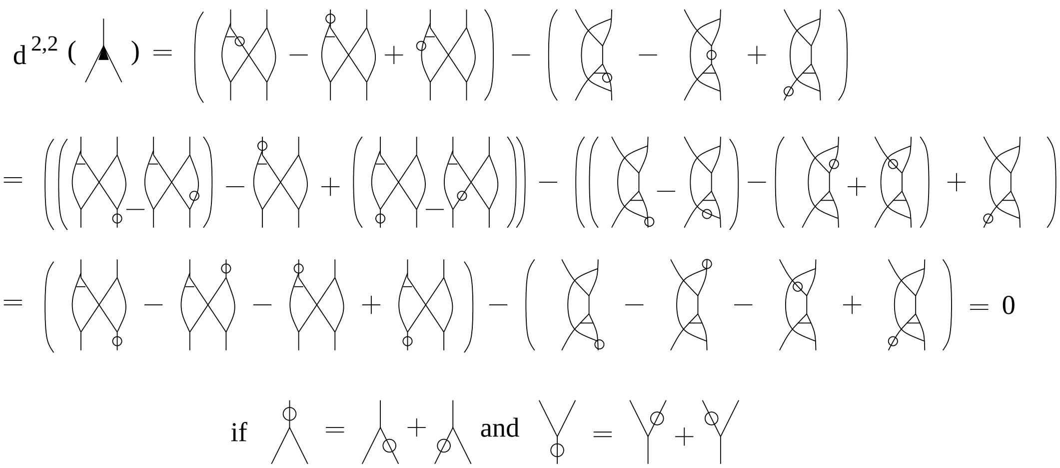

Figure 13: The 2-cocycle condition for a 2-coboundary, Part II

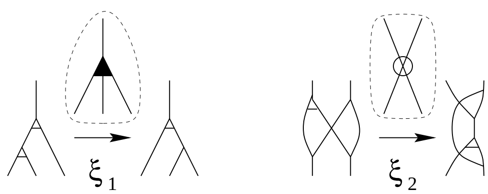

Figure 14: Diagrams for 3-cochains

By coassociativity applied to y and $\eta_1(x \otimes y_{(2)(2)})$, the second term is equal to

$$\eta_1(x \otimes y_{(2)})_{(1)} \otimes y_{(1)(1)}\eta_1(x \otimes y_{(2)})_{(2)(1)} \otimes y_{(1)(2)}\eta_1(x \otimes y_{(2)})_{(2)(2)},$$

which is equal, by compatibility, to

$$\eta_1(x \otimes y_{(2)})_{(1)} \otimes (y_{(1)}\eta_1(x \otimes y_{(2)})_{(2)})_{(1)} \otimes (y_{(1)}\eta_1(x \otimes y_{(2)})_{(2)})_{(2)}.$$

This last term is represented exactly by the last term in the second line of Fig. 20, and therefore is cancelled. The map represented by the second term in the second line of Fig. 20 cancels with the third term by coassociativity, and the fourth term cancels with the sixth by coassociativity applied twice and compatibility once. Other cases (Figs. 18, 19) are computed similarly. □

4.5 Cohomology Groups

For convenience define $C^0(H;H) = 0$, $D_0 = 0 : C^0(H;H) \to C^1(H;H)$.

Then Theorems 4.3, 4.5 are summarized as:

Theorem 4.6 *$\mathcal{C} = (C^n, D_n)_{n=0,1,2,3}$ is a chain complex.*

This enables us to define:

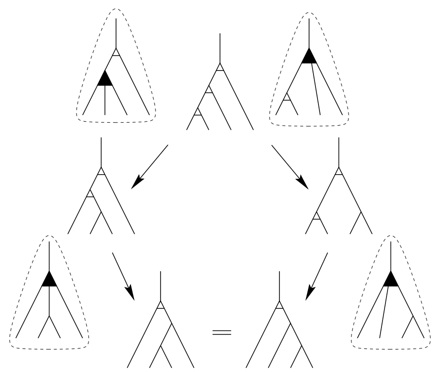

Figure 15: The first 3-differential $d^{3,1}$

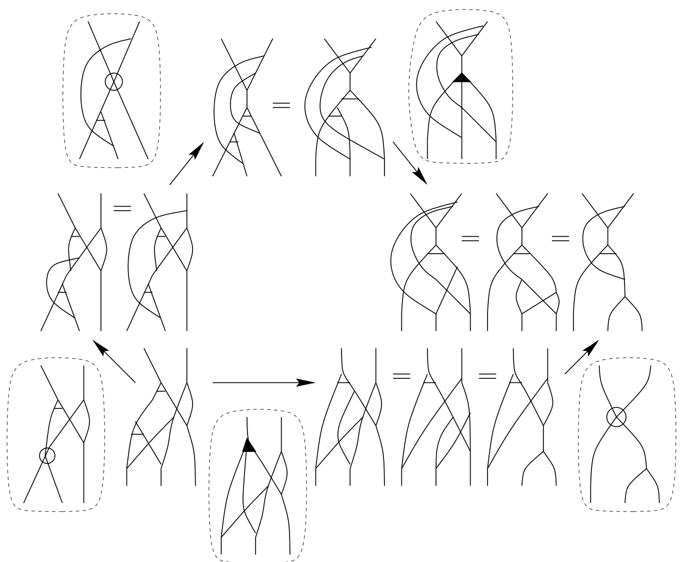

Figure 16: The second 3-differential $d^{3,2}$

Definition 4.7 The *adjoint* n-coboundary, cocycle and cohomology groups are defined by:

$$\begin{aligned} B^n(H;H) &= \text{Image}(D_{n-1}), \\ Z^n(H;H) &= \text{Ker}(D_n), \\ H^n(H;H) &= Z^n(H;H)/B^n(H;H) \end{aligned}$$

for $n = 1, 2, 3$.

5 Examples

5.1 Group Algebras

Let G be a group and $H = kG$ be its group algebra with the coefficient field k (char $k \neq 2$). Then H has a Hopf algebra structure induced from the group operation as multiplication, $\Delta(x) = x \otimes x$

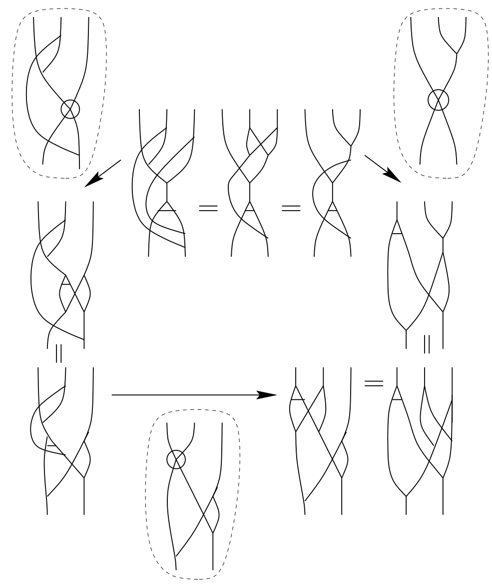

Figure 17: The third 3-differential $d^{3,3}$

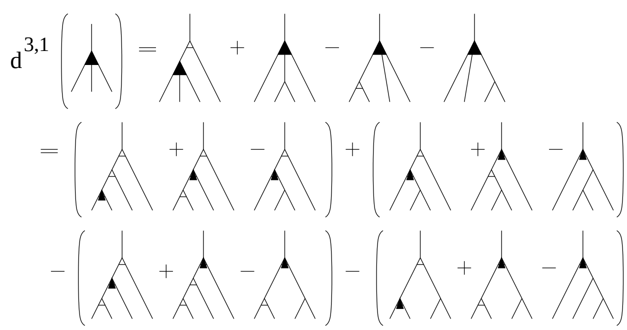

Figure 18: $d^{3,1}(d^{2,1}) = 0$

for basis elements $x \in G$, and the antipode induced from $S(x) = x^{-1}$ for $x \in G$. Here and below, we denote the conjugation action on a group G by $x \triangleleft y := y^{-1}xy$. Note that this defines a quandle structure on G; see [12].

Lemma 5.1 *$C^1_{\rm ad}(kG; kG) = 0$.*

Proof. For any given $w \in G$ write $f(w) = \sum_{u \in G} a_u(w)u$, where $a : G \to k$ is a function. Recall the definition

$$C^1_{\rm ad}(H; H) = \{ f \in \mathrm{Hom}_k(H, H) \mid f\mu = \mu(f \otimes 1) + \mu(1 \otimes f),\ \Delta f = (f \otimes 1)\Delta + (1 \otimes f)\Delta\ \}.$$

The LHS of the second condition is written as

$$\Delta f(w) = \Delta(\sum_u a_u(w)u) = \sum_u a_u(w) u \otimes u$$

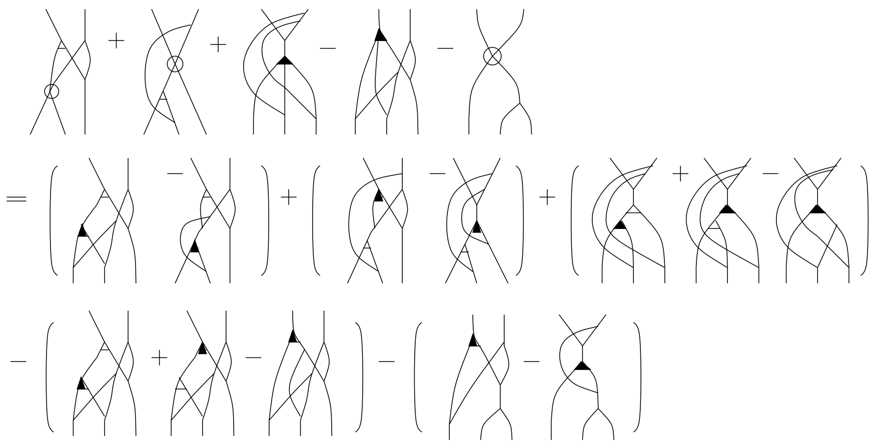

Figure 19: $d^{3,2}(d^{2,1}, d^{2,2}) = 0$

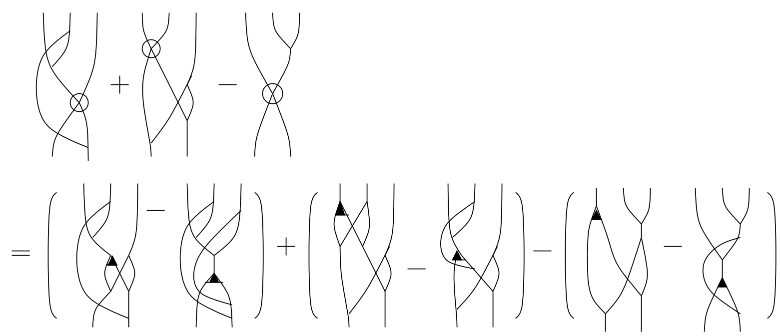

Figure 20: $d^{3,3}(d^{2,2}) = 0$

and the RHS is written as

$$((f \otimes 1)\Delta + (1 \otimes f)\Delta)(w) = ((f \otimes 1) + (1 \otimes f))(w \otimes w) = \sum_h a_h(w)(h \otimes w) + \sum_v a_v(w)(w \otimes v).$$

For a given w, fix u and then compare the coefficients of $u \otimes u$. In the LHS we have $a_u(w)$, while on the RHS $w = u$, and furthermore $w = h = v$ for $u \otimes u$. Thus the diagonal coefficient must satisfy $a_w(w) = a_w(w) + a_w(w)$, so that $a_w(w) = 0$ since char $k \neq 2$. In the case $w \neq u$, neither term of $h \otimes w$ nor $w \otimes v$ is equal to $u \otimes u$, hence $a_u(w) = 0$. $\square$

Lemma 5.2 *For $x, y \in G$, write $\phi(x, y) = \sum_u a_u(x, y)u$, where $a : G \times G \to k$. Then the induced linear map $\phi : kG \otimes kG \to kG$ is in $Z^2_{\mathrm{ad}}(kG; kG)$ if and only if a satisfies*

$$a_{x \lhd y}(x, y) + a_{(x \lhd y) \lhd z}(x \lhd y, z) - a_{(x \lhd y) \lhd z}(x, yz) = 0$$

for any $x, y, z \in G$.

Proof. The first 2-cocycle condition for $\phi : kG \otimes kG \to kG$ is written by:

$$z^{-1}\phi(x \otimes y)z + \phi(y^{-1}xy \otimes z) - \phi(x \otimes yz) = 0$$

for basis elements $x, y, z \in G$. The second is formulated by

$$\text{LHS} = \phi(x \otimes y) \otimes xy = \sum_u a_u(x,y)(u \otimes xy), \quad \text{RHS} = \sum_w a_w(x,y)(w \otimes yw).$$

They have the common term $u \otimes xy$ for $w = y^{-1}xy = u$, and otherwise they are different terms. Thus we obtain $a_w(x,y) = 0$ unless $w = y^{-1}xy$. For these terms, the first condition becomes

$$z^{-1}(a_{y^{-1}xy}(x,y)y^{-1}xy)z + a_{z^{-1}y^{-1}xyz}(y^{-1}xy, z)z^{-1}y^{-1}xyz - a_{z^{-1}y^{-1}xyz}(x, yz)z^{-1}y^{-1}xyz = 0$$

and the result follows. □

Remark 5.3 In the preceding proof, since the term $a_w(x,y) = 0$ unless $w = x \triangleleft y$, let $a_{x\triangleleft y}(x,y) = a(x,y)$. Then the condition stated becomes

$$a(x,y) + a(x \triangleleft y, z) - a(x, yz) = 0.$$

Proposition 5.4 *Let G be a group. Let $(\xi_1, \xi_2) \in C^3_{\text{ad}}(kG; kG)$, where ξ_1 is the map that is defined by linearly extending $\xi_1(x \otimes y \otimes z) = \sum_{u \in G} c_u(x,y,z)u$. Then $(\xi_1, \xi_2) \in Z^3_{\text{ad}}(kG; kG)$ if and only if $\xi_2 = 0$ and the coefficients satisfy the following properties:*

(a) $c_u(x,y,z) = 0$ if $u \neq z^{-1}y^{-1}xyz$ and

(b) $c(x,y,z) = c_{z^{-1}y^{-1}xyz}(x,y,z)$ satisfies

$$c(x,y,z) + c(x, yz, w) = c(y^{-1}xy, z, w) + c(x, y, zw).$$

Proof. Suppose $(\xi_1, \xi_2) \in Z^3_{\text{ad}}(kG; kG)$. Let ξ_2 be the map that is defined by linearly extending $\xi_2(x \otimes y) = \sum_{u,v \in G} a_{u,v}(x,y) u \otimes v$. Then the third 3-cocycle condition from Definition 4.4 gives: (abbreviating $a_{u,v}(x,y) = a_{u,v}$)

$$\begin{aligned} &d^{3,3}(\xi_1, \xi_2)(x \otimes y) \\ &\quad = \sum_{u_1,v_1} a_{u_1,v_1}(u_1 \otimes yu_1 \otimes v_1) + \sum_{u_2,v_2} a_{u_2,v_2}(u_2 \otimes v_2 \otimes xy) - \sum_{u_3,v_3} a_{u_3,v_3}(u_3 \otimes v_3 \otimes v_3) \quad = 0. \end{aligned}$$

We first consider terms in which the third tensorand is xy. From the third summand, this forces the second tensorand to be xy, so we collect the terms of the form $(u \otimes xy \otimes xy)$. This gives:

$$\sum_u (a_{u,xy} + a_{u,xy} - a_{u,xy})(u \otimes xy \otimes xy) = 0,$$

which implies $a_{u,xy} = 0$ for all $u \in G$. The remaining terms are

$$\sum_{u_1,v_1 \neq xy} a_{u_1,v_1}(u_1 \otimes yu_1 \otimes v_1) + \sum_{u_2,v_2 \neq xy} a_{u_2,v_2}(u_2 \otimes v_2 \otimes xy) - \sum_{u_3,v_3 \neq xy} a_{u_3,v_3}(u_3 \otimes v_3 \otimes v_3) = 0.$$

From the second sum we obtain $a_{u,v}(x,y) = 0$ for $v \neq xy$. In conclusion, if $d^{3,3}(\xi_1, \xi_2) = 0$ for kG then $\xi_2 = 0$.

We now consider $d^{3,2}(\xi_1, \xi_2)$, with $\xi_2 = 0$. Let ξ_1 be the map that is defined by linearly extending $\xi_1(x \otimes y \otimes z) = \sum_{u \in G} c_u(x,y,z)u$ for $x, y, z \in G$. The second 3-cocycle condition from Definition 4.4, with $\xi_2 = 0$, is $\sum_u c_u u \otimes yzu = \sum_v c_v v \otimes xyz$. In order to combine like terms, we need $yzu = xyz$, meaning $u = z^{-1}y^{-1}xyz$. Thus, $c_u(x,y,z) = 0$ except in the case when $u = z^{-1}y^{-1}xyz$. In this case, we obtain $\xi_1(x \otimes y \otimes z) = c(x,y,z)z^{-1}y^{-1}xyz \otimes xyz$ where $c(x,y,z) = c_{z^{-1}y^{-1}xyz}(x,y,z)$.

Finally we consider the first 3-cocycle condition from Definition 4.4, which is formulated for basis elements by

$$w^{-1}\ \xi_1(x \otimes y \otimes z)\ w + \xi_1(x \otimes yz \otimes w) = \xi_1(x \triangleleft y \otimes z \otimes w) + \xi_1(x \otimes y \otimes zw).$$

Substituting in the formula for $c(x,y,z)$ which we found above, we obtain

$$c(x,y,z) + c(x,yz,w) = c(y^{-1}xy, z, w) + c(x,y,zw).$$

This is a group 3-cocycle condition with the first term $x \cdot c(y,z,w)$ omitted. This is expected from Fig. 15. Constant functions, for example, satisfy this condition. $\square$

Next we look at a coboundary condition. A 3-coboundary is written as

$$\xi_1(x \otimes y \otimes z) = \sum_u c_u(x,y,z)u = d^{2,1}(\phi)(x \otimes y \otimes z) = z^{-1}\phi(x \otimes y)z + \phi(y^{-1}xy \otimes z) - \phi(x \otimes yz).$$

If we write $\phi(x,y) = \sum_u h_u(x,y)u$, then

$$\begin{aligned} &(d^{2,1}(\phi))(x \otimes y \otimes z) \\ &= z^{-1}\left(\sum_u h_u(x,y)u\right) z + \left(\sum_v h_v(y^{-1}xy, z)v\right) - \left(\sum_w h_w(x,yz)w\right) \\ &= \sum_g (\ h_{zgz^{-1}}(x,y) + h_g(y^{-1}xy, z) - h_g(x,yz)\ )\ g. \end{aligned}$$

Hence

$$c_u(x,y,z) = h_{zuz^{-1}}(x,y) + h_u(y^{-1}xy, z) - h_u(x,yz)$$

and in particular for the coefficients $c_u(x,y,z)$ from Proposition 5.4,

$$c(x,y,z) = c_{z^{-1}y^{-1}xyz}(x,y,z) = h_{y^{-1}xy}(x,y) + h_{z^{-1}y^{-1}xyz}(y^{-1}xy, z) - h_{z^{-1}y^{-1}xyz}(x,yz).$$

By setting $h_{y^{-1}xy}(x,y) = a(x,y)$, we obtain:

Lemma 5.5 *A 3-cocycle $c(x,y,z)$ is a coboundary if for some $a(x,y)$,*

$$c(x,y,z) = a(x,y) + a(y^{-1}xy, z) - a(x,yz).$$

Remark 5.6 From Remark 5.3, Proposition 5.4, and Lemma 5.5, we have the following situation. The 2-cocycle condition, the 3-cocycle condition, and the 3-coboundary condition, respectively, gives rise to the equations

$$\begin{aligned} &a(x,y) + a(y^{-1}xy, z) - a(x,yz) = 0, \\ &c(x,y,z) + c(x,yz,w) - c(y^{-1}xy, z, w) - c(x,y,zw) = 0, \\ &c(x,y,z) = a(x,y) + a(y^{-1}xy, z) - a(x,yz). \end{aligned}$$

This suggests a cohomology theory, which we investigate in Section 6.

Proposition 5.7 *For the symmetric group $G = S_3$ on three letters, we have $H^1_{\mathrm{ad}}(kG; kG) = 0$ and $H^2_{\mathrm{ad}}(kG; kG) \cong \bigoplus_3 (kG)$ for $k = \mathbb{C}$ and $\mathbb{F}_3$.*

Proof. By Lemma 5.1, we have $H^1_{\mathrm{ad}}(kG; kG) = 0$ and $B^2_{\mathrm{ad}}(kG, kG) = 0$. Hence $H^2(kG; kG) \cong Z^2_{\mathrm{ad}}(kG; kG)$, which is computed by solving the system of equations stated in Lemma 5.2 and Remark 5.3. Computations by *Maple* and *Mathematica* shows that the solution set is of dimension 3 and generated by $(a((1\ 2\ 3), (1\ 2))$, $a((2\ 3), (1\ 3\ 2))$, and $a((1\ 3), (1\ 2))$ for the above mentioned coefficient fields. □

5.2 Function Algebras on Groups

Let G be a finite group and k a field with $\mathrm{char}(k) \neq 2$. The set k^G of functions from G to k with pointwise addition and multiplication is a unital associative algebra. It has a Hopf algebra structure using $k^{G\times G} \cong k^G \otimes k^G$ with comultiplication defined through $\Delta : k^G \to k^{G\times G}$ by $\Delta(f)(u \otimes v) = f(uv)$ and the antipode by $S(f)(x) = f(x^{-1})$.

Now k^G has basis (the characteristic function) $\delta_g : G \to k$ defined by $\delta_g(x) = 1$ if $x = g$ and zero otherwise. Since $S(\delta_g) = \delta_{g^{-1}}$ and $\Delta(\delta_h) = \sum_{uv=h} \delta_u \otimes \delta_v$, the adjoint map becomes

$$\mathrm{ad}(\delta_g \otimes \delta_h) = \sum_{uv=h} \delta_{u^{-1}}\delta_g\delta_v = \begin{cases} \delta_g & \text{if } h = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.8 $C^1_{\mathrm{ad}}(k^G; k^G) = 0$.

Proof. Recall that

$$C^1_{\mathrm{ad}}(H; H) = \{f \in \mathrm{Hom}_k(H, H) \mid f\mu = \mu(f \otimes 1) + \mu(1 \otimes f),\ \Delta f = (f \otimes 1)\Delta + (1 \otimes f)\Delta\ \}.$$

Let $G = \{g_1, \dots, g_n\}$ be a given finite group and abbreviate $\delta_{g_i} = \delta_i$ for $i = 1, \dots, n$. Describe $f : k^G \to k^G$ by $f(\delta_i) = \sum_{j=1}^{n} s_i^j \delta_j$. Then $f\mu = \mu(f \otimes 1) + \mu(1 \otimes f)$ is written for basis elements by LHS $= f(\delta_i\delta_j)$ and

$$\begin{aligned} \text{RHS} &= f(\delta_i)\delta_j + \delta_i f(\delta_j) \\ &= (\sum_{\ell=1}^{n} s_i^\ell \delta_\ell)\delta_j + \delta_i(\sum_{h=1}^{n} s_j^h \delta_h) \\ &= s_i^j \delta_j + s_j^i \delta_i. \end{aligned}$$

For $i = j$ we obtain LHS$= \sum_{w=1}^{n} s_i^w \delta_w$ and RHS $= 2s_i^i\delta_i$ so that $s_i^j = 0$ for all i, j as desired. □

Lemma 5.9 $Z^2_{\mathrm{ad}}(k^G; k^G) = 0$.

Proof. Recall that $d^{2,1}(\eta_1) = \mathrm{ad}(\eta_1 \otimes 1) + \eta_1(\mathrm{ad} \otimes 1) - \eta_1(1 \otimes \mu)$ for $\eta_1 \in C^2_{\mathrm{ad}}(k^G, k^G)$. Describe

a general element $\eta_1 \in C^2_{\text{ad}}(k^G, k^G)$ by $\eta_1(\delta_i \otimes \delta_j) = \sum_\ell s^\ell_{i\,j}\delta_\ell$. Consider $d^{2,1}(\eta_1)(\delta_a \otimes \delta_b \otimes \delta_c)$. If $c \neq 1$, then the first term is zero by the definition of ad. If $c \neq 1$ and $b = 1$, then the third term is also zero, and we obtain that the second term $\eta_1(\delta_a \otimes \delta_c)$ is zero. Hence $\eta_1(\delta_a \otimes \delta_c) = 0$ unless $c = 1$. Next, set $b = c = 1$ in the general form. Then all three terms equal $\eta_1(\delta_a \otimes \delta_1)$ and we obtain $\eta_1(\delta_a \otimes \delta_1) = 0$, and the result follows. $\square$

By combining the above lemmas, we obtain the following:

Theorem 5.10 *For any finite group G and a field k, we have $H^n_{\text{ad}}(k^G; k^G) = 0$ for $n = 1, 2$.*

Observe that $k(G)$ and k^G are cohomologically distinct.

5.3 Bosonization of the Superline

Let H be generated by 1, g, x with relations $x^2 = 0$, $g^2 = 1$, $xg = -gx$ and Hopf algebra structure $\Delta(x) = x \otimes 1 + g \otimes x$, $\Delta(g) = g \otimes g$, $\epsilon(x) = 0$, $\epsilon(g) = 1$, $S(x) = -gx$, $S(g) = g$ (this Hopf algebra is called the bosonization of the superline [15], page 39, Example 2.1.7).

The operation ad is represented by the following table, where, for example, $\text{ad}(g \otimes x) = 2x$.

	1	g	x	gx
1	1	1	0	0
g	g	g	$2x$	$2x$
x	x	$-x$	0	0
gx	gx	$-gx$	0	0

Remark 5.11 The induced R-matrix R_{ad} has determinant 1, the characteristic polynomial is $(\lambda^2+1)^2(\lambda+1)^4(\lambda-1)^8$, and the minimal polynomial is $(\lambda^2+1)(\lambda+1)(\lambda-1)^2$.

Proposition 5.12 *The first cohomology of H is given by $H^1_{\text{ad}}(H, H) \cong k$.*

Proof. Recall that 1-cochains are given by

$$C^1_{\text{ad}}(H; H) = \{f \in \text{Hom}_k(H, H) \mid f\mu = \mu(f \otimes 1) + \mu(1 \otimes f),\ \Delta f = (f \otimes 1)\Delta + (1 \otimes f)\Delta\ \}.$$

Let $f \in C^1_{\text{ad}}(H; H)$. Assume that $f(x) = a + bx + cg + dxg$ and $f(g) = \alpha + \beta x + \gamma g + \delta xg$ where $a, b, c, d, \alpha, \beta, \gamma, \delta \in k$. Applying f to both sides of the equation $g^2 = 1$, one obtains $\alpha = \gamma = 0$. Similarly evaluating both sides of the equation $\Delta f = (f \otimes 1)\Delta + (1 \otimes f)\Delta$ at g gives $\beta = \delta = 0$, one obtains that $f(g) = 0$. In a similar way, applying f to the equations $x^2 = 0$ and $xg = -gx$ gives rise to, respectively, $a = 0$ and $c = 0$. Also evaluating $\Delta f = (f \otimes 1)\Delta + (1 \otimes f)\Delta$ at x gives rise to $d = 0$. We also have $f(x) = f(xg)g$ (since $g^2 = 1$),which implies $f(xg) = bxg$. In conclusion f satisfies $f(1) = 0 = f(g), f(x) = bx$, and $f(xg) = b(xg)$. Now consider f in the kernel of D_1, that is f satisfies

$$d^{1,1}(f) = \text{ad}(1 \otimes f) - f\text{ad} + \text{ad}(f \otimes 1).$$

It is directly checked on all the generators $u \otimes v$ of $H \otimes H$ that $d^{1,1}(f)(u \otimes v) = 0$. This implies that $H^1(H, H) \cong k$. $\square$

Proposition 5.13 *For any field k of characteristic not 2, $H^2_{\text{ad}}(H,H) \cong k^3$.*

Proof. With $d^{1,1} = 0$ from the preceding Proposition, we have $H^2_{\text{ad}}(H,H) \cong Z^2_{\text{ad}}(H,H)$.

For the convenience of the reader we compute, $\Delta(gx) = gx \otimes g + 1 \otimes gx$. A number of key facts will be repeatedly recalled; these are inclosed in boxes.

The first 2-differential is written as

$$\text{ad}(\phi(a \otimes b) \otimes c) + \phi(\text{ad}(a \otimes b) \otimes c) - \phi(a \otimes bc) = 0.$$

Take $b = c = 1$, then since $\text{ad}(a \otimes 1) = a$ for any $a \in H$, all three terms are the same and gives that $\boxed{\phi(a \otimes 1) = 0}$ for any a.

Take $a = g$ and $b = c = x$, then the third term vanishes and we obtain $\text{ad}(\phi(g \otimes x) \otimes x) + \phi(2x \otimes x) = 0$. For any possible value of $\phi(g \otimes x)$, the value of the first term is written as hx for some $h \in k$ from the table of ad above. Since ϕ is bilinear, constants can be renamed to obtain $\boxed{\phi(x \otimes x) = hx}$. A similar argument gives $\boxed{\phi(x \otimes gx) = h'x}$ from $a = g$, $b = x$ and $c = gx$, for another constant $h' \in k$.

The second differential is written as

$$\phi(a_{(1)} \otimes b_{(1)}) \otimes a_{(2)}b_{(2)} = \phi(a \otimes b_{(2)})_{(1)} \otimes b_{(1)}\phi(a \otimes b_{(2)})_{(2)}.$$

Taking $a = b = x$, we obtain the LHS $\phi(x \otimes x) \otimes 1 + (\phi(x \otimes g) + \phi(g \otimes x)) \otimes x$. The RHS is $\phi(x \otimes x)_{(1)} \otimes g\phi(x \otimes x)_{(2)}$, and using that $\phi(x \otimes x) = hx$, we obtain $h(x \otimes g + g \otimes gx)$ for the RHS. Since there is no $\otimes 1$ term in the RHS, we obtain $\phi(x \otimes x) = 0$, and in particular, $h = 0$, which makes RHS $= 0$, and we also obtain $\boxed{\phi(x \otimes g) = -\phi(g \otimes x)}$.

Let $a = b = g$ in the second differential. Then LHS $= \phi(g \otimes g) \otimes 1$ and RHS $= \phi(g \otimes g)_{(1)} \otimes g\phi(g \otimes g)_{(2)}$. This implies that $\phi(g \otimes g)$ is written as $h_g g$ for some $h_g \in k$, and RHS $= h_g(g \otimes g^2) = h_g(g \otimes 1) =$ LHS. With $a = b = c = g$ in the first differential, we obtain $\text{ad}(h_g g \otimes g) + h_g g - 0 = 0$, hence, in fact, $h_g = 0$ if 2 is invertible, giving rise to $\boxed{\phi(g \otimes g) = 0}$.

Let $a = g$ and $b = x$ in the second differential. Then the LHS $= \phi(g \otimes x) \otimes g$, and the RHS $= \phi(g \otimes x)_{(1)} \otimes g\phi(g \otimes x)_{(2)}$. For the RHS to have terms ending in $\otimes g$ only, $\phi(g \otimes x)$ can have neither g nor gx terms since they would result in a ($\otimes 1$) term, so let $\phi(g \otimes x) = h_{g,x}1 + h'_{g,x}x$. Then one computes RHS $= (h_{g,x}1 + h'_{g,x}x) \otimes g + h'_{g,x}(g \otimes gx)$. Equating this with LHS, we obtain $h'_{g,x} = 0$. Thus we obtained $\boxed{\phi(g \otimes x) = h_{g,x}1 = -\phi(x \otimes g)}$. In the first differential, take $a = b = g$ and $c = x$ to obtain $\boxed{\phi(g \otimes gx) = \phi(g \otimes x) = h_{g,x}1}$.

Let $a = 1$ and $b = x$ in the second differential. Then the LHS $= \phi(1 \otimes x) \otimes 1 + \phi(1 \otimes g) \otimes x$, and the RHS $= \phi(1 \otimes x)_{(1)} \otimes g\phi(1 \otimes x)_{(2)}$. For the RHS to have terms ending in $\otimes 1$ or $\otimes x$ only, $\phi(1 \otimes x)$ can have neither 1 nor x terms since they would result in a ($\otimes g$) term, so let $\boxed{\phi(1 \otimes x) = h_{1,x}g + h_{1,g}gx}$. Then one computes RHS $= h_{1,x}g \otimes 1 + h_{1,g}(gx \otimes 1 + 1 \otimes x)$. Comparing with the LHS, we obtain $\boxed{\phi(1 \otimes g) = h_{1,g}1}$. With $a = 1$, $b = x$ and $c = g$ in the first differential, we also obtain $\boxed{\phi(1 \otimes gx) = -h_{1,x}g + h_{1,g}gx}$.

Recall that $\phi(x \otimes gx) = h'x$. For $a = x$ and $b = gx$ in the second differential gives

$$\begin{aligned} \text{LHS} &= \phi(x \otimes gx) \otimes g - \phi(g \otimes gx) \otimes gx = h'(x \otimes g) - h_{g,x}(1 \otimes gx) \\ \text{RHS} &= -h_{g,x}(1 \otimes gx) + h'(x \otimes 1 + g \otimes x) \end{aligned}$$

which implies $\boxed{\phi(x \otimes gx) = 0}$.

In the second differential, take $a = gx$ and $b = x$. Then we obtain

$$\begin{aligned} \text{LHS} &= \phi(gx \otimes x) \otimes g + (\phi(gx \otimes g) + \phi(1 \otimes x)) \otimes gx \\ &= \phi(gx \otimes x) \otimes g + (\phi(gx \otimes g) + h_{1,x}g + h_{1,g}gx) \otimes gx \\ \text{RHS} &= \phi(gx \otimes x)_{(1)} \otimes g\phi(gx \otimes x)_{(2)}. \end{aligned}$$

The LHS has only $\otimes g$ and $\otimes gx$ terms, so that $\phi(gx \otimes x)$ does not have g or gx terms, and we can write $\phi(gx \otimes x) = h_{gx,x}1 + h'_{gx,x}x$ and compute RHS $= h_{gx,x}(1 \otimes g) + h'_{gx,x}(x \otimes g + g \otimes gx)$. Comparing with the LHS we obtain $h'_{gx,x}g = \phi(gx \otimes g) + h_{1,x}g + h_{1,g}gx$, so that $\boxed{\phi(gx \otimes g) = (h'_{gx,x} - h_{1,x})g - h_{1,g}gx}$.

By the first differential with $(a, b, c) = (gx, g, x)$, we obtain

$$\phi(gx \otimes gx) = 2(h'_{gx,x} - h_{1,x})x - (h_{gx,x}1 + h'_{gx,x}x) = -h_{gx,x}1 + (h'_{gx,x} - 2h_{1,x})x.$$

By the first differential with $(a, b, c) = (gx, gx, g)$, we obtain

$$(-h_{gx,x}1 - (h'_{gx,x} - 2h_{1,x})x) + 0 + (h_{gx,x}1 + h'_{gx,x}x) = 0$$

which implies $h_{1,x} = 0$. In particular, we obtain $\boxed{\phi(gx \otimes g) = h'_{gx,x}g - h_{1,g}gx}$ and $\boxed{\phi(gx \otimes gx) = -h_{gx,x}1 + h'_{gx,x}x}$. By the second differential with $a = b = gx$, we obtain

$$\begin{aligned} \text{LHS} &= \phi(gx \otimes gx) \otimes 1 + \phi(1 \otimes gx) \otimes (gx)g \\ &= (-h_{gx,x}1 + h'_{gx,x}x) \otimes 1 - h_{1,g}(gx \otimes x) \\ \text{RHS} &= \phi(gx \otimes g)_{(1)} \otimes (gx)\phi(gx \otimes g)_{(2)} + \phi(gx \otimes gx)_{(1)} \otimes \phi(gx \otimes gx)_{(2)} \\ &= (h'_{gx,x}(g \otimes gxg) + h_{1,g}(gx \otimes x)) + (-h_{gx,x}(1 \otimes 1) + h'_{gx,x}(x \otimes 1 + g \otimes x)) \end{aligned}$$

and comparing the terms we obtain $2h_{1,g} = 0$. In summary, resetting free variables by $h_{g,x} = \alpha$, $h_{gx,x} = \beta$ and $h'_{gx,x} = \gamma$, we obtained a general solution represented by the following table.

	1	g	x	gx
1	0	0	0	0
g	0	0	$\alpha 1$	$\alpha 1$
x	0	$-\alpha 1$	0	0
gx	0	γg	$\beta 1 + \gamma x$	$-\beta 1 + \gamma x$

It is checked, either by hand, or computer guided calculations, that these are indeed solutions. $\square$

6 Adjoint, Groupoid, and Quandle Cohomology Theories

From Remark 5.6, the adjoint cohomology leads us to cohomology, especially for conjugate groupoids of groups as defined below. Through the relation between Reidemeister moves for knots and the adjoint, groupoid cohomology, we obtain a new construction of quandle cocycles. In this section we investigate these relations. First we formulate a general definition. Many formulations of groupoid

cohomology can be found in literature, and relations of the following formulation to previously known theories are not clear. See [20], for example.

Let $\mathcal{G}$ be a groupoid with objects $\mathrm{Ob}(\mathcal{G})$ and morphisms $G(x,y)$ for $x,y \in \mathrm{Ob}(\mathcal{G})$. Let $f_i \in G(x_i, x_{i+1})$, $0 \le i < n$, for non-negative integers i and n. Let $C_n(\mathcal{G})$ be the free abelian group generated by

$$\{(x_0, f_0, \dots, f_n) \mid x_0 \in \mathrm{Ob}(\mathcal{G}), f_i \in G(x_i, x_{i+1}), 0 \le i < n\}.$$

The boundary map $\partial : C_{n+1}(\mathcal{G}) \to C_n(\mathcal{G})$ is defined by by linearly extending

$$\begin{aligned}\partial(x_0, f_0, \dots, f_n) &= (x_1, f_1, \dots, f_n)\\ &+ \sum_{i=0}^{n-1}(-1)^{i+1}(x_0, f_0, \dots, f_{i-1}, f_i f_{i+1}, f_{i+2}, \dots, f_n)\\ &+ (-1)^{n+1}(x_0, f_0, \dots, f_{n-1}).\end{aligned}$$

Then it is easily seen that this differential defines a chain complex.

The corresponding groupoid 1- and 2-cocycle conditions are written as:

$$\begin{aligned}&a(x_1, f_1) - a(x_0, f_0 f_1) + a(x_0, f_0) = 0\\ &c(x_1, f_1, f_2) - c(x_0, f_0 f_1, f_2) + c(x_0, f_0, f_1 f_2) - c(x_0, f_0, f_1) = 0\end{aligned}$$

The general cohomological theory of homomorphisms and extensions applies, such as:

Remark 6.1 Let $\mathcal{G}$ be a groupoid and A be an abelian group regarded as a one-object groupoid. Then $\alpha : \hom(x_0, x_1) \to A$ gives a groupoid homomorphism from $\mathcal{G}$ to A, which sends $\mathrm{Ob}(\mathcal{G})$ to the single object of A, if and only if $a : C_1(\mathcal{G}) \to A$, defined by $a(x_0, f_0) = \alpha(f_0)$, is a groupoid 1-cocycle.

Next we consider extensions of groupoids. Define $\circ : (\hom(x_0, x_1) \times A) \times (\hom(x_1, x_2) \times A) \to \hom(x_0, x_2) \times A$ by

$$(f_0, a) \circ (f_1, b) = (f_0 f_1, a + b + c(x_0, f_0, f_1))$$

where $c(x_0, f_0, f_1) \in \hom(C_2(\mathcal{G}), A)$. If $\mathcal{G} \times A$ is a groupoid, the function c with the value $c(x_0, f_0, f_1)$ is a groupoid 2-cocycle.

Example 6.2 Let G be a group. Define the *conjugate groupoid* of G, denoted $\widehat{G}$, by:

$$\begin{aligned}\mathrm{Ob}(\widehat{G}) &= G\\ \mathrm{Mor}(\widehat{G}) &= G \times G\end{aligned}$$

where the source of the morphism $(x, y) \in \hom(x, y^{-1}xy)$ is x and its target is $y^{-1}xy$, for $x, y \in G$. Composition is defined by $(x, y) \circ (y^{-1}xy, z) = (x, yz)$. For this example, the groupoid 1- and 2-cocycle conditions are:

$$\begin{aligned}&a(x, y) + a(y^{-1}xy, z) - a(x, yz) = 0,\\ &c(x, y, z) + c(x, yz, w) - c(y^{-1}xy, z, w) - c(x, y, zw) = 0.\end{aligned}$$

Diagrammatic representations of these equations are depicted in Figs. 21, 22. Furthermore, c is a coboundary if

$$c(x, y, z) = a(x, y) + a(y^{-1}xy, z) - a(x, yz).$$

Compare with Remark 5.6.

For $G = \mathbb{S}_3$, the symmetric group on 3 letters, with coefficient group $\mathbb{C}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$ and $\mathbb{Z}_7$, respectively, the dimensions of the conjugation groupoid 2-cocycles are 3, 5, 4, 3 and 3.

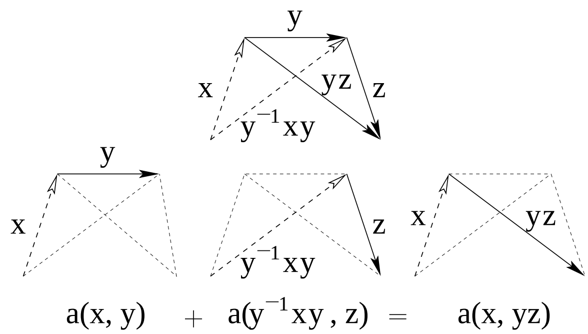

Figure 21: Diagrams for a groupoid 1-cocycle

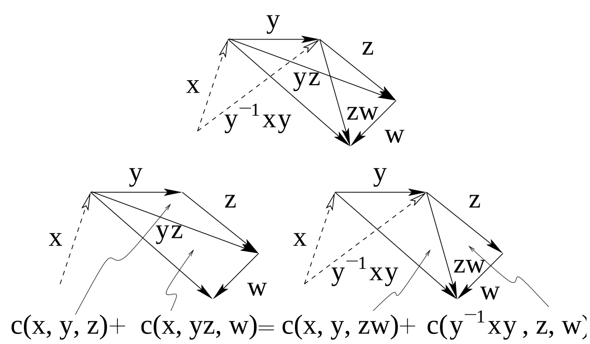

Figure 22: Diagrams for a groupoid 2-cocycle

For the rest of the section, we present new constructions of quandle cocycles from groupoid cocycles of conjugate groupoids of groups. Let G be a finite group, and $a : G^2 \to k$ be adjoint 2-cocycle coefficients that were defined in Remark 5.3. These satisfy

$$a(x, y) + a(x \triangleleft y, z) - a(x, yz) = 0.$$

Proposition 6.3 *Let $\psi(x, y) = a(x, y)$. Then ψ satisfies the rack 2-cocycle condition*

$$\psi(x, y) + \psi(x \triangleleft y, z) = \psi(x, z) + \psi(x \triangleleft z, y \triangleleft z).$$

Proof. By definition $\psi(x, y) + \psi(x \triangleleft y, z) = a(x, yz)$, and $\psi(x, z) + \psi(x \triangleleft z, y \triangleleft z) = a(x, z) + a(z^{-1}xz, z^{-1}yz) = a(x, z(y \triangleleft z))$. $\square$

Let G be a finite group, and $c : G^3 \to k$ be a coefficient of the adjoint 3-cocycle defined in Proposition 5.4. This satisfies

$$c(x, y, z) + c(x, yz, w) = c(x \triangleleft y, z, w) + c(x, y, zw).$$

Proposition 6.4 *Let G be a group that is considered as a quandle under conjugation. Then $\theta : G^3 \to k$ defined by $\theta(x,y,z) = c(x,y,z) - c(x,z,z^{-1}yz)$ is a rack 3-cocycle.*

Proof. We must show that θ satsifies

$$\begin{aligned} & \theta(x,y,z) + \theta(x \triangleleft z, y \triangleleft z, w) + \theta(x,z,w) \\ = \quad & \theta(x \triangleleft y, z, w) + \theta(x,y,w) + \theta(x \triangleleft w, y \triangleleft w, z \triangleleft w). \end{aligned}$$

We compute

$$\begin{aligned} \text{LHS} - \text{RHS} \quad = \quad & [\ c(x,y,z) - c(x,z,z^{-1}yz)\ ] \\ & +[\ c(z^{-1}xz, z^{-1}yz, w) - c(z^{-1}xz, w, w^{-1}z^{-1}yzw)\ ] \\ & +[\ c(x,z,w) - c(x,w,w^{-1}zw)\ ] \\ & -[\ c(y^{-1}xy, z, w) - c(y^{-1}xy, w, w^{-1}zw)\ ] \\ & -[\ c(x,y,w) - c(x,w,w^{-1}yw)\ ] \\ & -[\ c(w^{-1}xw, w^{-1}yw, w^{-1}zw) - c(w^{-1}xw, w^{-1}zw, w^{-1}z^{-1}yzw)\ ] \\ = \quad & [\ c(x,y,z) - c(y^{-1}xy, z, w)\ ] \\ & -[\ c(x,z,z^{-1}yz) - c(z^{-1}xz, z^{-1}yz, w)\ ] \\ & +[\ c(x,z,w) - c(z^{-1}xz, w, w^{-1}z^{-1}yzw)\ ] \\ & -[\ c(x,w,w^{-1}zw) - c(w^{-1}xw, w^{-1}zw, w^{-1}z^{-1}yzw)\ ] \\ & -[\ c(x,y,w) - c(y^{-1}xy, w, w^{-1}zw)\ ] \\ & +[\ c(x,w,w^{-1}yw) - c(w^{-1}xw, w^{-1}yw, w^{-1}zw)\ ] \\ = \quad & [\ -c(x,yz,w) + c(x,y,zw)\ ] \\ & -[\ -c(x,zz^{-1}yz,w) + c(x,z,z^{-1}yzw)\ ] \\ & +[\ -c(x,zw,w^{-1}z^{-1}yzw) + c(x,z,ww^{-1}z^{-1}yzw)\ ] \\ & -[\ -c(x,ww^{-1}zw,w^{-1}z^{-1}yzw) + c(x,w,w^{-1}zww^{-1}yw)\ ] \\ & -[\ -c(x,yw,w^{-1}zw) + c(x,y,ww^{-1}zw)\ ] \\ & +[\ -c(x,ww^{-1}yw,w^{-1}zw) + c(x,w,w^{-1}yww^{-1}zw)\ ] \\ = \quad & 0 \end{aligned}$$

as desired. $\square$

7 Deformations of R-matrices by adjoint 2-cocycles

In this section we give, in an explicit form, deformations of R-matrices by 2-cocycles of the adjoint cohomology theory we developed in this paper. Let H be a Hopf algebra and ad its adjoint map. In Section 3 a deformation of (H, ad) was defined to be a pair (H_t, ad_t) where H_t is a $k[[t]]$-Hopf algebra given by $H_t = H \otimes k[[t]]$ with all Hopf algebra structures inherited by extending those on H_t. Let $A = (H \otimes k[[t]])/(t^2))$ and the Hopf algebra structure maps $\mu, \Delta, \epsilon, \eta, S$ be inherited on A. As a vector space A can be regarded as $H \oplus tH$

Recall that a solution to the YBE, R-matrix R_{ad} is induced from the adjoint map. Then from the constructions of the adjoint cohomology from the point of view of the deformation theory, we obtain the following deformation of this R-matrix induced from the adjoint map.

Theorem 7.1 *Let $\phi \in Z^2_{\rm ad}(H;H)$ be an adjoint 2-cocycle. Then the map $R : A \otimes A \to A \otimes A$ defined by $R = R_{{\rm ad}+t\phi}$ satisfies the YBE.*

Proof. The equalities of Lemma 3.2 hold in the quotient $A = (H \otimes k[[t]])/(t^2)$, where $n = 1$ and the modulus t^2 is considered. These cocycle conditions, on the other hand, were formulated from the motivation from Lemma 2.2 for the induced R-matrix $R_{\rm ad}$ to satisfy the YBE. Hence these two lemmas imply the theorem. □

Example 7.2 In Subsection 5.3, the adjoint map ad was computed for the bosonization H of the superline, with basis $\{1, g, x, gx\}$, as well as a general 2-cocycle ϕ with three free variables α, β, γ written by $\phi(g \otimes x) = \phi(g \otimes gx) = \alpha 1$, $\phi(x \otimes g) = -\alpha 1$, $\phi(gx \otimes g) = \gamma g$, $\phi(gx \otimes x) = \beta 1 + \gamma x$, $\phi(gx \otimes gx) = -\beta 1 + \gamma x$, and zero otherwise. Thus we obtain the deformed solution to the YBE $R = R_{{\rm ad}+t\phi}$ on A with three variables $t\alpha, t\beta, t\gamma$ of degree one.

8 Concluding Remarks

In [7] we concluded with *A Compendium of Questions* regarding our discoveries. Here we attempt to address some of these questions by providing relationships between this paper and [7], and offer further questions for our future consideration.

It was pointed out in [7] that there was a clear distinction between the Hopf algebra case and the cocommutative coalgebra case as to why self-adjoint maps satisfy the YBE. In [7] a cohomology theory was constructed for the coalgebra case. In this paper, many of the same ideas and techniques, in particular deformations and diagrams, were used to construct a cohomology theory in the Hopf algebra case, with applications to the YBE and quandle cohomology.

The aspects that unify these two theories are deformations and a systematic process we call "diagrammatic infiltration." So far, these techniques have only been successful in defining coboundaries up through dimension 3. This is a deficit of the diagrammatic approach, but diagrams give direct applications to other algebraic problems such as the YBE and quandle cohomology, and suggest further applications to knot theory. By taking the trace as in Turaev's [21], for example, a new deformed version of a given invariant is expected to be obtained.

Many questions remain: Can 3-cocycles be used for solving the tetrahedral equation? Can they be used for knotted surface invariants? Can the coboundary maps be expressed skein theoretically? How are the deformations of R-matrices related to deformations of underlying Hopf algebras? When a Hopf algebra contains a coalgebra, such as the universal enveloping algebra and its Lie algebra together with the ground field of degree-zero part, what is the relation between the two theories developed in this paper and in [7]? How these theories, other than the same diagrammatic techniques, can be uniformly formulated, and to higher dimensions?

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